10-Mar L11

Monday, March 10, 2025 1:45 PM

§3.1 cont'd

ex. describe image of $T(\vec{x}) = A\vec{x}$ $\mathbb{R}^2 \to \mathbb{R}^2$ given $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} 1x_1 + 3x_2 \\ 2x_1 + 6x_2 \end{matrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix} x_2$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2$$

vectors are scalars and ||

$$= (x_1 + 3x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are parallel, im(T) is the line of <u>all</u> scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

ex. describe image of linear transformation (LT) $T(\vec{x}) = A\vec{x}$ $\mathbb{R}^2 \to \mathbb{R}^3$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

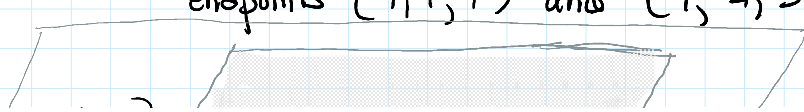
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x_2$$

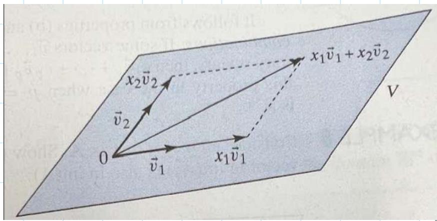
i.e. all linear combinations of the column vectors in A $\rightarrow$ image

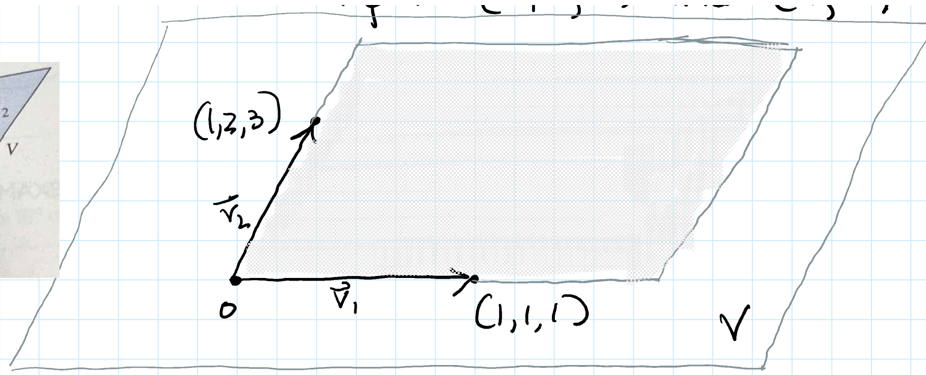
let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

conclusion: im(T) is plane V spanned $\vec{v}_1$ and $\vec{v}_2$

↳ the plane through the origin and endpoints (1, 1, 1) and (1, 2, 3)







def'n: $\text{span}(\vec{v}_1, \ldots, \vec{v}_m) = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m$ where $c_1, \ldots, c_m \in \mathbb{R}$

i.e. set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

image of a LT $T(\vec{x}) = A\vec{x}$ is the span of column vectors of A

given: $T: \mathbb{R}^m \to \mathbb{R}^n$

following properties hold:

- $\vec{0}$ (zero vector) in $\mathbb{R}^n$ is in $\text{im}(T)$
- $\text{im}(T)$ is closed under addition (i.e., $\vec{v}_1$ and $\vec{v}_2$ are in $\text{im}(T)$ then so is $\vec{v}_1 + \vec{v}_2$)
- $\text{im}(T)$ is closed under scalar multiplication (i.e. if $\vec{v}$ is in $\text{im}(T)$ and k is a scalar then $k\vec{v}$ is in $\text{im}(T)$ as well)

Kernel / Null Space

Comparable to finding zeros of a polynomial function where: $f(x) = 0$

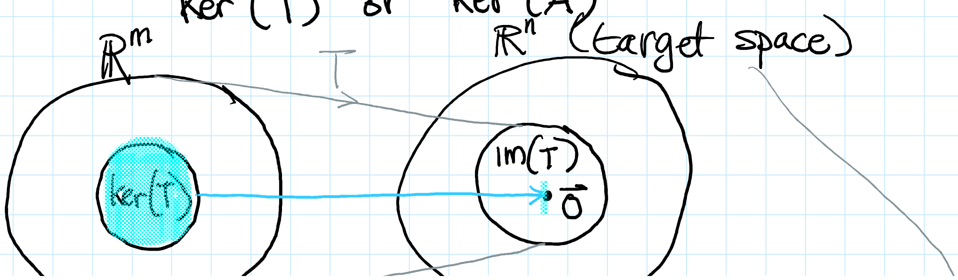
similarly, zeros of a LT are of interest

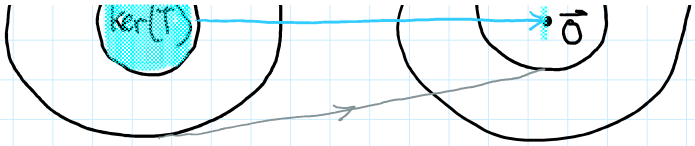
given $T(\vec{x}) = A\vec{x}$ $\mathbb{R}^m \to \mathbb{R}^n$

then its kernel $:=$ all zeros of LT s.t. $T(\vec{x}) = A\vec{x} = \vec{0}$

$\downarrow$ denoted as

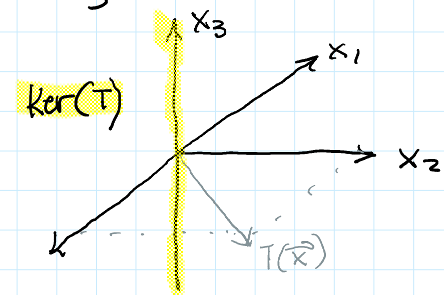
$\ker(T)$ or $\ker(A)$





$\text{im}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^m\}$ is a subset of $\mathbb{R}^n$ of T
and
$\ker(T) = \{\vec{x} \in \mathbb{R}^m \text{ s.t. } T(\vec{x}) = \vec{0}\}$ is a subset of $\mathbb{R}^m$ (domain) of T

revisit: given LT $\mathbb{R}^3 \to \mathbb{R}^3$ that projected $\vec{x}$ orthogonally into x_1-x_2 plane
ker(T) is the set of solutions for $T(\vec{x}) = \vec{0}$
i.e. set of vectors whose orthogonal projection onto x_1-x_2 plane is zero, which are vectors on x_3-axis
i.e. all scalars of $\vec{e}_3$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



ex. find kernel of LT of $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\vec{x}$ $\mathbb{R}^3 \to \mathbb{R}^2$

solve $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left| \begin{array}{l} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array} \right| \longrightarrow \begin{array}{l} x_1 = x_3 \\ x_2 = -2x_3 \end{array}$$

let x_3 = arbitrary t

then
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Conclusion: ker(T) is line spanned by $\begin{bmatrix} 1 \\ -2 \\ ? \end{bmatrix}$ in $\mathbb{R}^3$

ex. find kernel of $T(\vec{x}) = A\vec{x}$ $\quad \mathbb{R}^5 \to \mathbb{R}^4$ where $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & 7 \end{bmatrix}$ changed / was correct

again, solve $A\vec{x} = \vec{0}$

$$\text{rref}[A \mid \vec{0}] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left| \begin{array}{l} x_1 + 2x_2 + 3x_4 - 4x_5 = 0 \\ x_3 - 4x_4 + 5x_5 = 0 \end{array} \right|$$

$$\left| \begin{array}{l} x_1 = -2x_2 - 3x_4 + 4x_5 \\ x_3 = 4x_4 - 5x_5 \end{array} \right|$$

let $\quad x_2 = r \quad x_4 = s \quad x_5 = t$

arbitrary constants

then solutions are vectors in the form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \to \ker(T) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}\right)$$

Kernel Properties given $T: \mathbb{R}^m \to \mathbb{R}^n$

- $\vec{0}$ in $\mathbb{R}^m \in \ker(T)$
- kernel is closed under addition
- kernel is closed under scalar multiplication

ex. given A is an invertible, $n \times n$ matrix

recall that, since A is invertible, then, if $\vec{b} = \vec{0}$ then $A\vec{x} = \vec{0}$ has a unique solution: $\vec{x} = \vec{0}$

since A is invertible and its only solution is $\vec{x} = \vec{0}$ then $\ker(A) = \{\vec{0}\}$

tying concepts together.

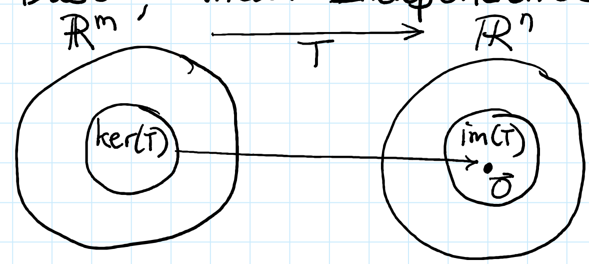
given $n \times m$ matrix A $\quad$ • $\ker(A) = \{\vec{0}\}$ iff $\text{rank}(A) = m$

- if $\ker(A) = \{\vec{0}\}$ then $m \leq n$
- if $m > n$ $\exists$ non zero-vectors in $\ker(A)$

given $n \times n$ matrix A, $\ker(A) = \{\vec{0}\}$ iff A is invertible
(square)

§3.2 Subspaces, Bases, Linear Independence

recall:



kernel is subset of $\mathbb{R}^m$

Image is subset of $\mathbb{R}^n$

Subspace of $\mathbb{R}^n$

a subset W of vector space $\mathbb{R}^n$ is called a (linear) <u>subspace</u> of $\mathbb{R}^n$ if the following properties hold:

1. W contains $\vec{0}$ in $\mathbb{R}^n$
2. W is closed under addition
3. W is closed under scalar multiplication

collectively, can say that W is closed under <u>linear combination</u>